Nonlinear Analysis of a Cantilever Beam

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The General Rotorcraft Aeromechanical Stability Program (GRASP) was developed to analyze the steady-state and linearized dynamic behavior of rotorcraft in hovering and axial flight conditions. Because of the nature of problems GRASP was created to solve, the geometrically nonlinear behavior of beams is one area in which the program must perform well in order to be of any value. Numerical results obtained from GRASP are compared to both static and dynamic experimental data obtained for a cantilever beam undergoing large displacements and rotations caused by deformation. The correlation is excellent in all cases.

Nomenclature

b_i^P = basis vectors at the deformed beam tip
b_i^R = basis vectors at the beam root
b = beam breadth
C_{ij} = direction cosine matrix relating the beam root and tip
e = error in beam cross-sectional measurement
E = modulus of elasticity
I_2 = geometrical cross-sectional property
I_4 = geometrical cross-sectional property
p = projection of b_1^P onto the $b_1^R - b_2^R$ plane
t = beam thickness
U = strain energy
α = material nonlinearity coefficient
β = torsional deflection that was experimentally measured
κ = curvature of the beam
ϕ_i = Rodrigues parameters
θ = load angle

Introduction

THE General Rotorcraft Aeromechanical Stability Program (GRASP)[1] is capable of treating the nonlinear static and linearized dynamic behavior of structures represented by collections of rigid-body and beam elements that may be connected in an arbitrary fashion and are permitted to have large relative motions.

GRASP combines multibody and finite-element technology, having taken the strong points from each area and integrated them together into a single, comprehensive package. GRASP differs from standard multibody programs by considering flexible-body and aeroelastic effects, including simple, nonlinear, unsteady aerodynamics. GRASP differs from standard finite-element programs by allowing multiple levels of substructures in which the substructures can move and/or rotate relative to others with no small-angle approximations. An overview of the features of GRASP as a program can be found in Ref. 1; details of the analysis are given in Ref. 2. The theoretical basis of the multibody/finite-element aspects of the analysis are addressed in companion papers.[3,4]

Because of the nature of the problems GRASP was created to solve, the geometrically nonlinear behavior of beams is one area in which the program must perform well in order to be of value. The main structural element in GRASP is the aeroelastic beam, a geometrically nonlinear beam element based on the kinematics, internal, and inertial forces of Ref. 5 and the aerodynamics of Ref. 6.

This paper is to present numerical results from GRASP for comparison with static and dynamic experimental data for large deflections of an end-loaded cantilevered beam. Details of the experiment may be found in Refs. 7 and 8. GRASP results are presented along with results from previous analyses that are shown for comparison. Although the present results do not exercise many of the features and power of GRASP, they do serve to validate much of the code dealing with the beam element's ability to model highly nonlinear behavior.

Experiment

An experiment done at Princeton University[7,8] (under Aeroflightdynamics Directorate sponsorship) was selected as a test case with which to validate GRASP. This experiment consisted of measuring the static deformation and fundamental flatwise and edgewise natural frequencies of a uniform, nonrotating, cantilever beam with a mass attached to the tip (Fig. 1). The beam was slender and sufficiently flexible to undergo large displacements (still at small strains) due to the presence of the tip mass. Beam load angle and mass of the tip weight were varied throughout the appropriate ranges.

The beam was fabricated from 7075 aluminum. The mass density was assumed to be 2.626×10^{-4} lb-s^2/in.^{4} The length of the beam was measured to be 19.985 in. The thickness and width of the beam were measured at 0.1251 in. and 0.4999 in., respectively. Assuming a gravitational constant equal to 386.089 in./s^2, the mass per unit length was determined to be 1.6424×10^{-5} lb-s^2/in.^{2} The mass moments of inertia were 2.1420×10^{-8} lb-s^2 (flatwise) and 3.4204×10^{-7} lb-s^2 (edgewise).

Determination of the appropriate values of bending stiffness proved to be more difficult. Both static and dynamic predicted behavior are sensitive to the value of the stiffnesses;

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therefore, stiffnesses were determined as accurately as possible. Attempted inference of equivalent beam properties from classical linear formulas for deflection vs load for the two uncoupled cases, load angles of 0 and 90 deg where all deformation is planar, yield contradictory information – even when only small deflections are considered. At a load angle of 0 deg (the edgewise-bending case), linear theory is too stiff. But, at a load angle of 90 deg (the flatwise-bending case) linear theory is too soft. With the failure of linear theory, we turned to a simple, nonlinear, planar elastica model[9] to attempt to arrive at independent values for equivalent beam stiffnesses. To allow for other effects not present in the simple elastica model, an extra parameter in the form of a material nonlinearity coefficient α was introduced. With the assumption that the beam is inextensible, the strain energy is expressed as

$$U = E/2\,(I_2\kappa^2 + \alpha I_4\kappa^4/2) \tag{1}$$

where κ is the planar curvature of the beam and I_2 and I_4 are geometrical cross-sectional properties defined as follows. For edgewise deflection

$$I_2 = b^3t/12 \tag{2a}$$

$$I_4 = b^5t/80 \tag{2b}$$

and for flatwise deflection

$$I_2 = bt^3/12 \tag{3a}$$

$$I_4 = bt^5/80 \tag{3b}$$

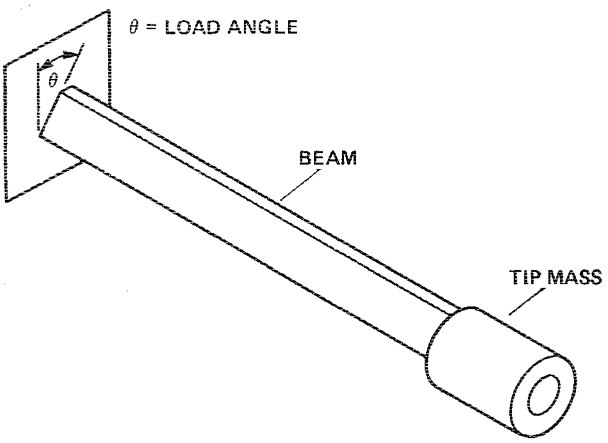


Fig. 1 Schematic of the Princeton beam experimental apparatus.

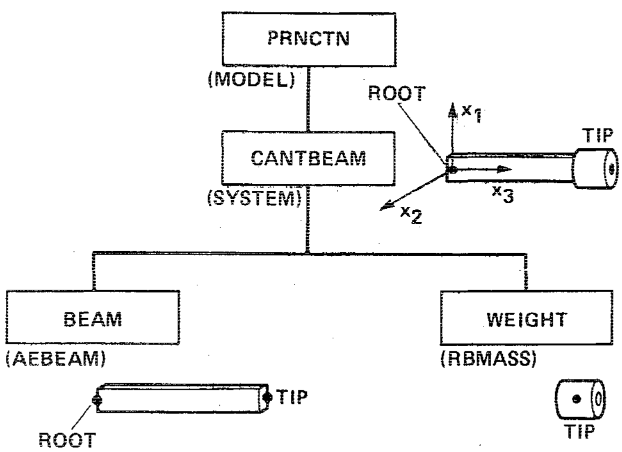


Fig. 2 Hierarchical GRASP model of the Princeton beam.

where b and t are the breadth and thickness dimensions, respectively, of the cross section of the beam. Considering only the uncoupled (i.e., planar) static deflections (load angles of 0 and 90 deg), an equation for the deflection of the tip of the beam was derived as a function of the beam bending stiffness and other unknowns. Once the equation for the tip deflection was derived, then a nonlinear least-squares method was used to determine the best EI_2 and α that fit the experimental data for both uncoupled flatwise and uncoupled edgewise deflections. The value obtained for α was ignored since GRASP does not consider material nonlinearity. The two values of E inferred from the bending stiffnesses and the cross-section geometry were averaged and multiplied by the cross-sectional area to obtain the axial stiffness. A value of Poisson's ratio equal to 0.31 was assumed, and the shear modulus G was inferred from E. The following stiffnesses resulted:

$$\begin{aligned} \text{axial stiffness} &= 6.2856 \times 10^5 \text{ lb} \\ \text{flatwise stiffness} &= 8.4487 \times 10^2 \text{ lb-in.}^2 \\ \text{edgewise stiffness} &= 1.2689 \times 10^4 \text{ lb-in.}^2 \\ \text{torsional stiffness} &= 1.0538 \times 10^3 \text{ lb-in.}^2 \end{aligned}$$

The moments of inertia of the various tip masses used were estimated with some gross assumptions, since these values are relatively unimportant. The only properties of the tip masses stated in Refs. 7 and 8 were the masses. There was also a photograph depicting a tip mass with a hollow cylindrical shape. With this information in mind, several assumptions were made: the density of the tip mass was that of steel 0.284 lb/in.^{3}), the inner radius was 0.375 in., and, finally, the length of the tip mass was equal to its outer diameter. With these assumptions, the moments of inertia for tip masses of varying size were calculated and are given in Table 1.

GRASP Model

The GRASP model for the Princeton experiment is depicted in Fig. 2. Subsystem PRNCTN, the *model-type* subsystem generated internally by GRASP, represents the complete structure. The first child of CANTBEAM is an *aeroelastic beam* element named BEAM. An *aeroelastic beam connectivity* constraint associates the element's root and tip nodes with the nodes ROOT and TIP in the subsystem CANTBEAM. The definition of the element includes specifying the orders of the polynomials used to represent the displacements. The typical approach in finite-element programs would be to use several elements with the transverse displacements approximated by cubic polynomials and the axial displacement and torsion approximated by linear polynomials. Instead, for this analysis we use one element with eighth-order polynomials for bending and sixth-order polynomials for axial displacement and torsion. This yields a total of 32 element degrees of freedom (six of which are constrained out by the clamped-end condition). Essentially the same results are obtained when the order of each polynomial is reduced by one.

Table 1 Estimated inertial properties of tip mass

Weight (lb)	Lateral moments of inertia (lb-in.s^2)	Axial moments of inertia (lb-in.s^2)
1.0	1.0822×10^{-3}	8.2356×10^{-4}
2.0	3.3676×10^{-3}	2.6784×10^{-3}
3.0	6.5673×10^{-3}	5.3169×10^{-3}
4.0	1.0561×10^{-2}	8.6363×10^{-3}
5.0	1.5276×10^{-2}	1.2573×10^{-2}
6.0	2.0658×10^{-2}	1.7083×10^{-2}
7.0	2.6670×10^{-2}	2.2131×10^{-2}
8.0	3.3278×10^{-2}	2.7691×10^{-2}
9.0	4.0457×10^{-2}	3.3741×10^{-2}
10.0	4.8185×10^{-2}	4.0261×10^{-2}
10.42	5.1589×10^{-2}	4.3135×10^{-2}
10.46	5.1919×10^{-2}	4.3413×10^{-2}

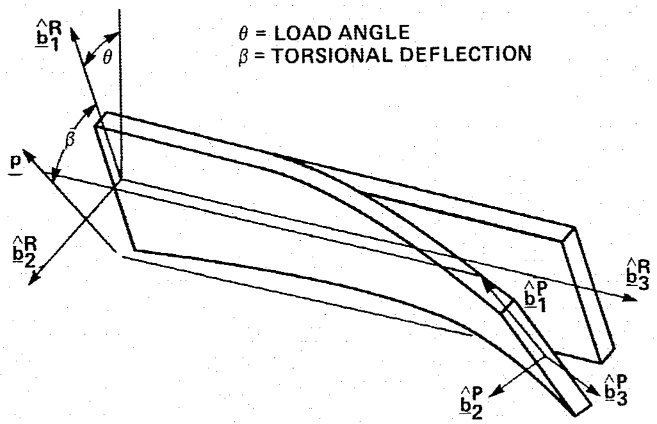
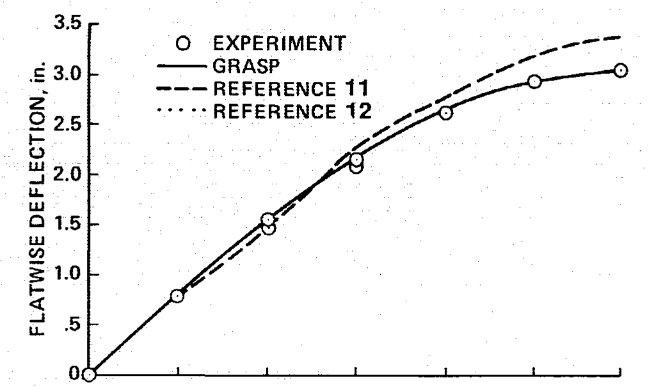
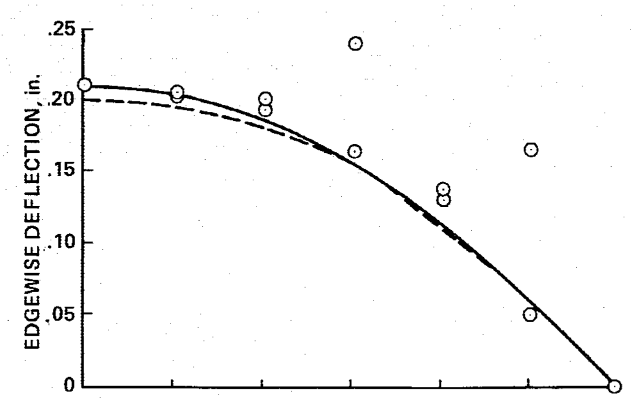


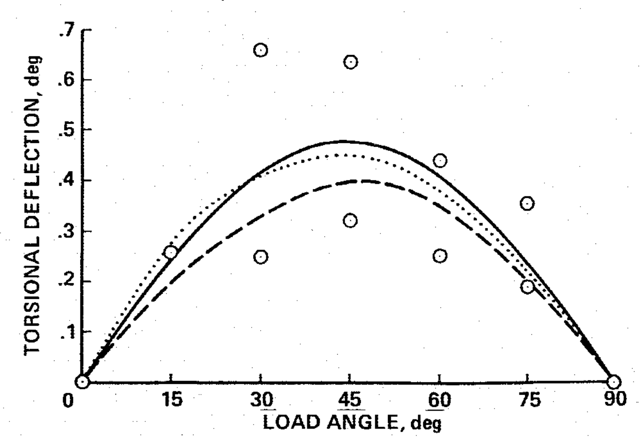
Fig. 3 Schematic of undeformed and deformed beams showing b_i^R (along the fixed axes at the root), b_i^P (along the principal axes at the tip), p (the projection of b_1^P onto the plane determined by b_1^R and b_2^R), and β (the angle measured in the experiment).



a) Flatwise vs load angle



b) Edgewise vs load angle



c) Torsional vs load angle

Fig. 4 GRASP correlation with the Princeton experiment (1-lb tip mass): static deflections vs load angle.

Correlation of GRASP Results With Experiment

GRASP expresses static rotations in terms of Rodrigues parameters,[2,10] and so a minor amount of postprocessing is needed to convert the GRASP output to the projected angle β as measured in Ref. 7 (Fig. 3). Consider the orthogonal triad at the root of the beam that remains aligned with the principal axes at the root. Introduce a dextral triad of unit vectors associated with those axes denoted by b_i^R for $i = 1,2$, and 3. Now consider a similar dextral triad at the tip of the beam, denoted by b_i^P for $i = 1,2$, and 3, where the deflections and rotations were measured in the experiments. The relationship between the triads is simply

$$b_i^P = C_{ij} b_j^R \tag{4}$$

where a repeated index implies summation over its range. A line along the breadth of the cross section is then aligned with $b_1^P = C_{1i} b_i^R$. Now consider the projection of b_1^P in the plane determined by b_1^R and b_2^R denoted by p. The expression for p can be easily determined as

$$p = b_1^P - b_1^P \cdot b_3^R\, b_3^R \tag{5a}$$

$$= C_{11} b_1^R + C_{12} b_2^R \tag{5b}$$

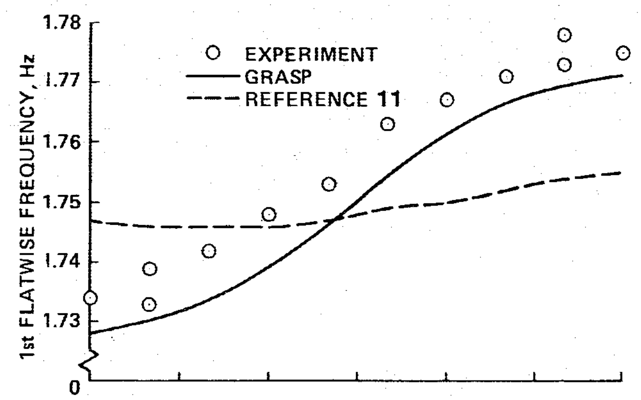
The angle measured in the experiments is the angle between p and b_1^R. From Fig. 3, it is clear that

$$\beta = \sin^{-1}\left(C_{12}/\sqrt{1 - C_{13}{}^2}\right) \tag{6}$$

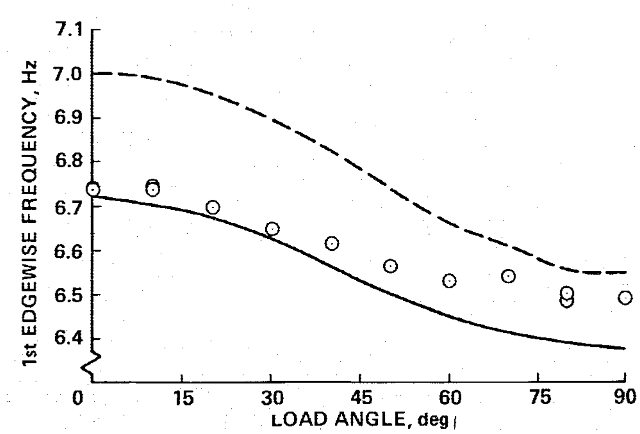
In terms of Rodrigues parameters

$$C_{12} = \frac{\phi_3 + (\phi_1\phi_2/2)}{1 + (\phi_1^2 + \phi_2^2 + \phi_3^2)/4} \tag{7a}$$

$$C_{13} = \frac{-\phi_2 + (\phi_1\phi_3/2)}{1 + (\phi_1^2 + \phi_2^2 + \phi_3^2)/4} \tag{7b}$$



a) First flatwise



b) First edgewise frequencies vs load angle

Fig. 5 GRASP correlation with the Princeton experiment (1-lb tip mass).

where ϕ_1, ϕ_2, and ϕ_3 are the Rodrigues parameters associated with the rotation of the tip node. (It should be noted that the matrix of direction cosines in this work is the transpose of the one in Ref. 10, and that the Rodrigues parameters used in GRASP differ from those of Ref. 10 by a factor of 2.)

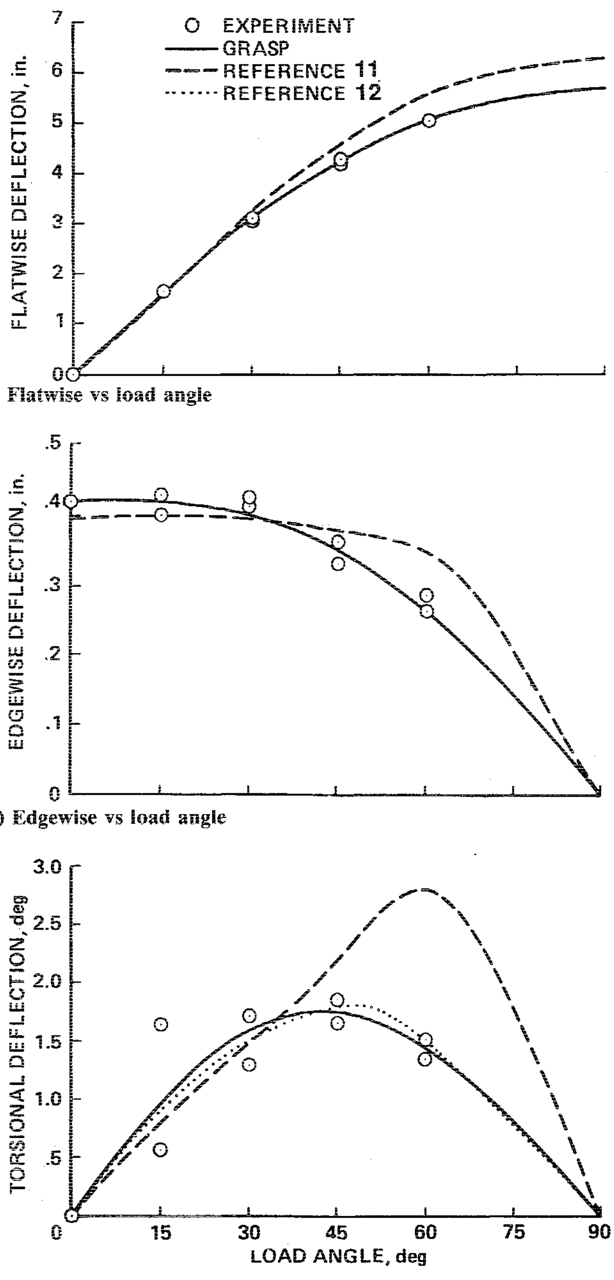
Also, all GRASP deflections have the deflections for no tip mass subtracted out before the results are plotted with the experimental data. All frequencies calculated by GRASP were converted from rad/s to Hz.

The results from GRASP and the experimental data are also compared to those of two previous works.[11,12] The calculations presented in Ref. 11 are based on the equations of Ref. 13, which is restricted to moderate rotations caused by deformation. The moderate rotation approximation requires that the squares of the rotational components are small compared to unity and is evidenced in the equations of Ref. 11 primarily as a truncation of each component of the curvature vector to second-degree nonlinearity.

The equations of Ref. 12 are specialized for the type of structure used in the experiment. Although they are essentially identical to those of Ref. 11, certain terms of third degree in the unknowns are retained in the components of the curvature vector, based on the observation that the coefficients of those terms are relatively large. The coefficients are only large because of the relatively large ratio of breadth to thickness of the Princeton beam cross section. The derivatives of these components appear in the equilibrium equations, and their coefficients depend on the relative magnitudes of the effective bending and torsional rigidities. These added terms would *not* be appropriate if the cross-section geometry were such that the bending and torsional stiffnesses were of the same order of magnitude.

In contrast, the components of the curvature vector in the GRASP equations are not truncated but are retained in their complete form except for the approximation of small extensional strain of the elastic axis. It is important to note that the equations in the GRASP analysis do not require that terms be added or removed in this manner. This is an important consideration for general-purpose analyses, the equations for which should not have need of alteration merely because of changes in properties.

First, results are presented for a 1-lb tip mass. Figure 4 shows the static deflections vs load angle. The GRASP correlation for flatwise and edgewise is excellent. Results from Refs. 11 and 12 are shown here for comparison, with results from Ref. 12 presented only for the torsional deflections. Transverse displacements from Ref. 12 were only available for load angles of 30 and 40 deg and, therefore, a complete load angle sweep could not be shown. It should be noted, however, that the transverse displacement results from Ref. 12 agrees well both with experiment and with GRASP. Reference 11 is not as accurate throughout the entire range of load angle as GRASP. For torsional deflection, the GRASP calculations cut right through the middle of the experimental scatter. The experimental scatter here is so large, however, that it is difficult to say which curve best fits the data.

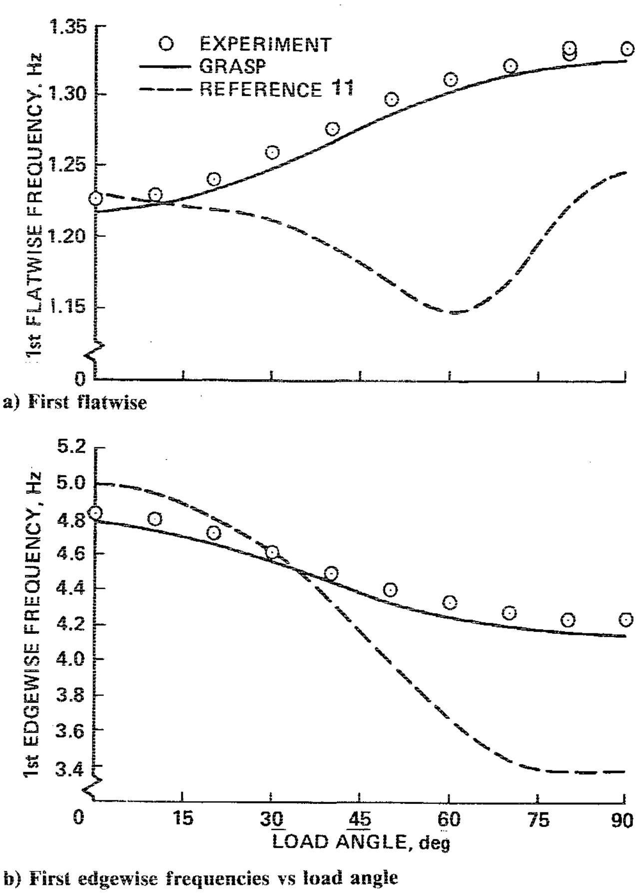


a) Flatwise vs load angle

b) Edgewise vs load angle

c) Torsional vs load angle

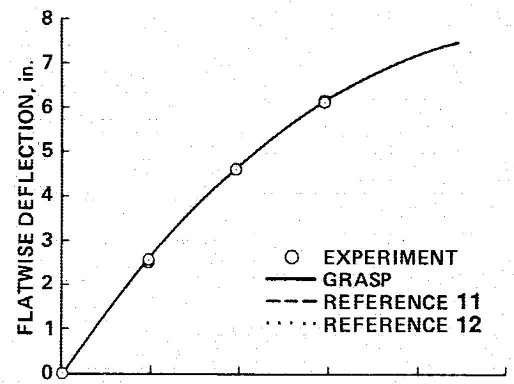
Fig. 6 GRASP correlation with the Princeton experiment (2-lb tip mass): static deflections vs load angle.



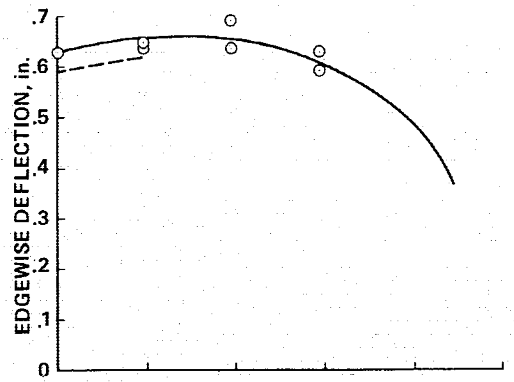
a) First flatwise

b) First edgewise frequencies vs load angle

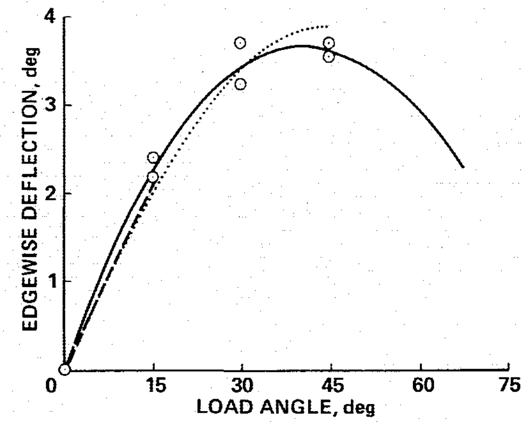
Fig. 7 GRASP correlation with the Princeton experiment (2-lb tip mass).



a) Flatwise vs load angle

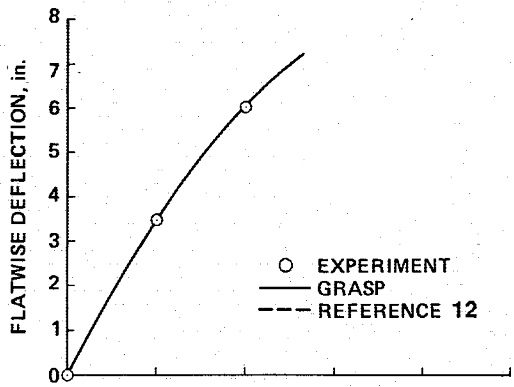


b) Edgewise vs load angle

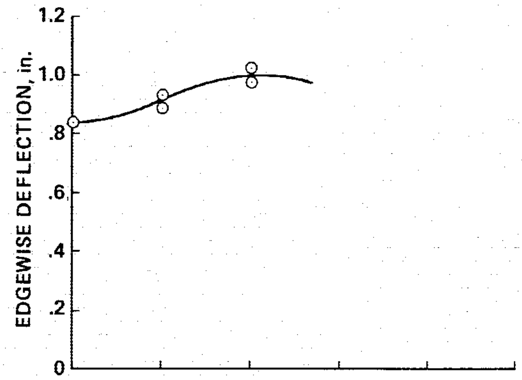


c) Torsional vs load angle

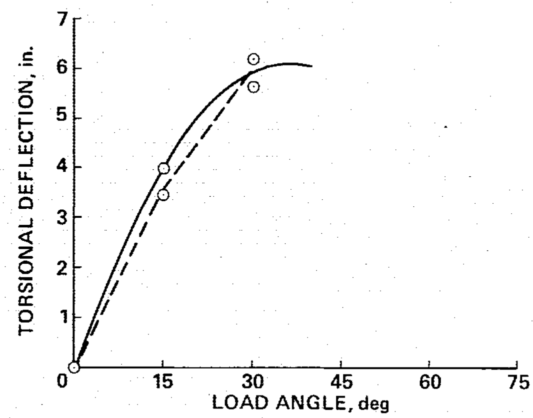
Fig. 8 GRASP correlation with the Princeton experiment (3-lb tip mass): static deflections vs load angle.



a) Flatwise vs load angle

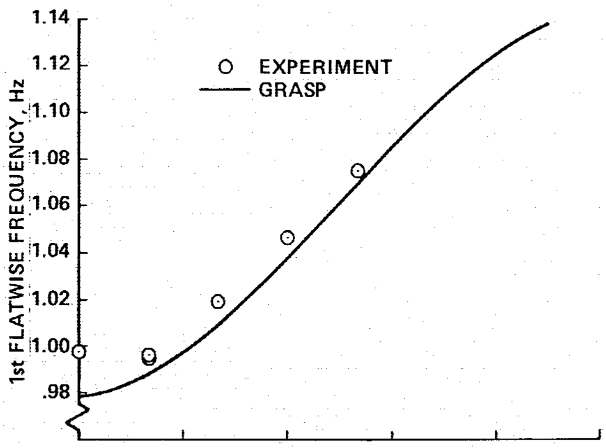


b) Edgewise vs load angle

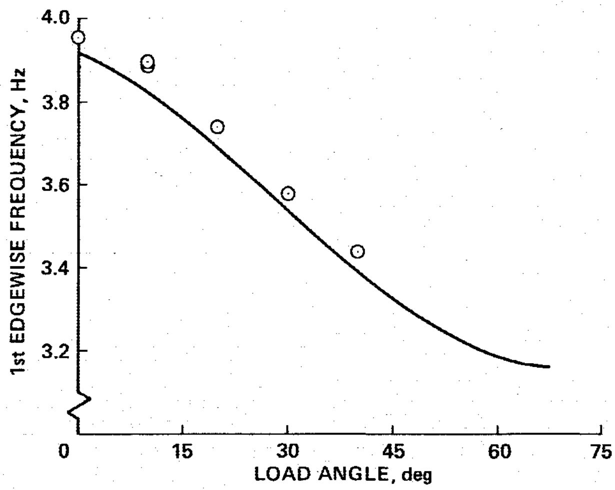


c) Torsional vs load angle

Fig. 10 GRASP correlation with the Princeton experiment (4-lb tip mass): static deflections vs load angle.

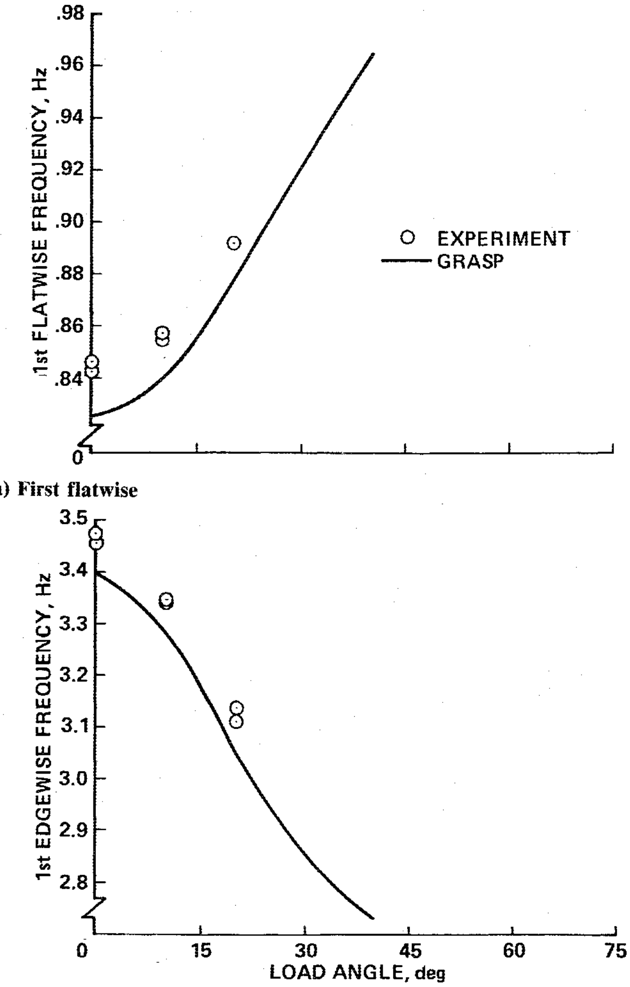


a) First flatwise



b) First edgewise frequencies vs load angle

Fig. 9 GRASP correlation with the Princeton experiment (3-lb tip mass).



a) First flatwise

b) First edgewise frequencies vs load angle

Fig. 11 GRASP correlation with the Princeton experiment (4-lb tip mass).

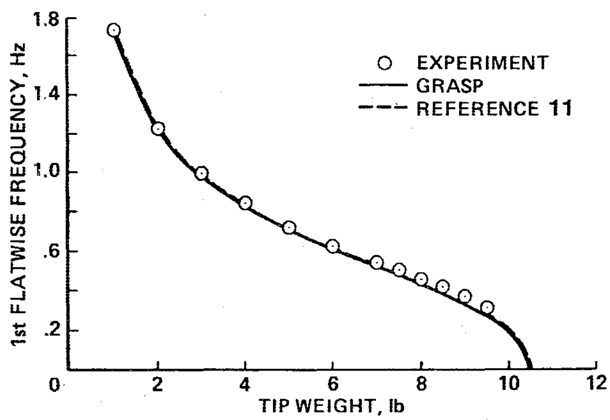


Fig. 12 GRASP correlation with the Princeton experiment: first flatwise frequency vs weight of tip mass.

Figure 5 displays the flatwise and edgewise frequencies vs load angle. The GRASP results are only slightly offset from the experimental values and follow the trend exactly. The average error is approximately 0.5%. Reference 11 follows the trend for the edgewise frequency but does not pick up the trend for the flatwise frequency. It is interesting to note that the moderate rotation theory is showing signs of failing even when the rotations are still moderate. Reference 12 does not consider the dynamics.

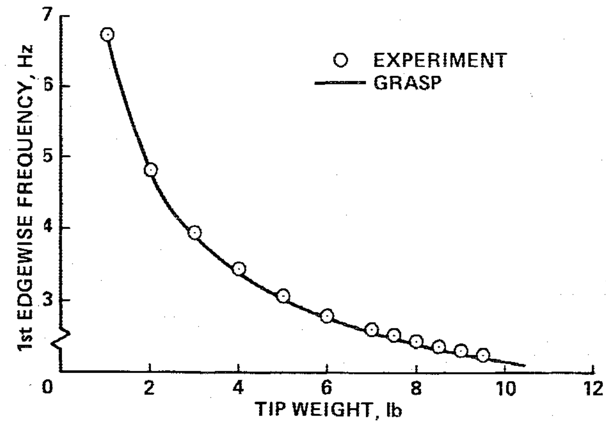


Fig. 13 GRASP correlation with the Princeton experiment: first edgewise frequency vs weight of tip mass.

The 2-lb tip-mass results are presented next. Here, the torsional data have much less scatter than in the 1-lb case. Again, GRASP correlates excellently with the static deflection, as shown in Fig. 6. Also again, Ref. 11 is close but tends to deviate through certain portions of the load angle sweep. This deviation from the data is large for load angles above 40 deg. Reference 12, however, correlates quite well with the static data.

Figure 7 shows the flatwise and edgewise frequencies. The GRASP predictions are again slightly low for both of the frequencies but follows the trends very nicely. Reference 11, although matching the data fairly well at 0 deg load angle, strays from the data at higher load angles.

Figure 8 presents the static results from the 3-lb tip-mass case. All three analyses appear to match the data well over the range shown. However, Ref. 11 results are available for only a load angle up to 15 deg whereas Ref. 12 calculated results up to 45 deg. The failure of Ref. 11 to converge past 15 deg is probably a result of the restriction to moderate rotation in that analysis. The third-degree terms retained in Ref. 12 allow its results to be obtained further, but still not as far as that of GRASP. GRASP converges out to 67.5 deg and fails because the initial guess of all generalized coordinates being equal to zero is too far away from the final answer. At the time the results were generated, GRASP did not possess the capability to have an initial guess input.

In Fig. 9, the flatwise and edgewise frequency sweeps are shown for the experimental data and the GRASP analysis. The correlation is excellent. For the 4-lb tip-mass case, there was not a very large experimental sweep available because of the large displacements that the beam underwent. GRASP did, however, correlate very well in the statics (Fig. 10) and the dynamics (Fig. 11). Reference 12 also correlates well with the static torsional deflection data

Figures 12 and 13 summarize flatwise and edgewise frequency vs tip mass. The lateral buckling load can be inferred from Fig. 12. As the tip mass increases, the flatwise frequency will tend toward zero. When lateral buckling is imminent, the frequency will drop to zero very rapidly, as indicated by both GRASP and Ref. 11. The edgewise frequency shown in Fig. 13 does not reach zero since there is no buckling in this mode.

Concluding Remarks

GRASP is a general-purpose program with both the detail and the generality to accurately model the end-loaded cantilever beam, as presented herein, as well or better than the special-purpose analyses in Refs. 11 and 12. Although this experiment demonstrated significant nonlinear behavior both statically and dynamically, GRASP accurately predicts the results. The equations upon which GRASP is based are not restricted as far as the magnitudes of displacement or rotation.

Only the strains are required to be small compared to unity. The GRASP analysis is shown herein to be valid for these types of problems. As noted in the introduction, however, the present validation does not exercise many of the capabilities of the program. Also, as pointed out in Ref. 1, the analysis does need to be extended to treat beams for which shear deformation would be important.

Acknowledgement

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